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ISSN 1333-1124
eISSN 1849-1391

A SHEAR LOCKING-FREE BEAM FINITE ELEMENT BASED ON THE MODIFIED TIMOSHENKO BEAM THEORY

UDC 534-16

Summary

An outline of the Timoshenko beam theory is presented. Two differential equations of motion in terms of deflection and cross-section rotation are comprised in one equation and analytical expressions for displacements and sectional forces are given. Two different displacement fields are recognized, i.e. flexural and axial shear, and a modified beam theory with extension is worked out. Flexural and axial shear locking-free beam finite elements are developed. Reliability of the finite elements is demonstrated with numerical examples for a simply supported, clamped and free beam by comparing the obtained results with analytical solutions.

Key words: *Timoshenko beam theory, modified beam theory, flexural vibrations, axial shear vibrations, beam finite element, shear locking*

1. Introduction

Beam is used as a structural element in many engineering structures like frames and grillages. Also, the whole structure can be modelled as a beam to some extent, e.g. ship hulls, floating airports, etc. The Euler-Bernoulli theory is widely used for the simulation of slender beam behaviour. The theory for thick beam was extended by Timoshenko [1] in order to take the effect of shear into account. The shear effect is extremely strong in higher vibration modes due to the reduced mode half wave length.

The Timoshenko beam theory deals with two differential equations of motion in terms of deflection and cross-section rotation. Most papers use this theory, while a possibility to use only one equation in terms of deflection has been recognized recently [2,3].

The Timoshenko beam theory has come into focus with the development of the finite element method and its application in practice. A large number of finite elements have been worked out in the last decades [4-13]. They differ in the choice of interpolation functions for a mathematical description of deflection and rotation. The application of equal order polynomials leads to the so-called shear locking since the bending strain energy for a thin beam vanishes before the shear strain energy [6,11]. Various approaches have been developed in order to overcome this problem, but a unique solution has not been found yet [11]. The problem is partially solved by some approaches such as the Reduced Integration Element

(RIE), the Consistent Interpolation Element (CIE), and the Interdependent Interpolation Element (IIE), which are described in [12,13,14].

The above short description of the state of the art has motivated further investigation into this challenging problem. First, a physical aspect of the Timoshenko beam theory is analysed. It is found that actually two different displacement fields are hidden in the beam deflection and rotation, i.e. pure bending with transverse shear on one hand, and axial shear on the other. The latter is analogous to the bar stretching on an axial elastic support. The cross-section rotation and the axial shear slope are treated in the Timoshenko beam theory as one variable since they have the same stiffness in the analytical formulation. Understanding of the beam dynamic behaviour makes the development of modified beam theory possible, which would be as exact as the Timoshenko beam theory.

Based on the modified Timoshenko theory, a two-node beam finite element is developed by taking a static solution for interpolation functions. Also, a beam element is derived for axial shear vibrations. Both elements are shear locking-free. Illustrative examples are given and the obtained results are compared with those obtained in an analytical way.

2. Timoshenko beam theory

2.1 Basic equations

The Timoshenko beam theory deals with the beam deflection and angle of rotation of cross-section, w and ψ, respectively [1]. The sectional forces, i.e. bending moment and shear force, read

$$M = D\frac{\partial \psi}{\partial x}, \qquad Q = S\left(\frac{\partial w}{\partial x} + \psi\right), \tag{1}$$

where $D=EI$ is the flexural rigidity and $S=kGA$ is the shear rigidity, A is the cross-section area and I is its moment of inertia, k is the shear coefficient, and E is Young's modulus and $G = E/\left(2\left(1+\nu\right)\right)$ is the shear modulus. The value of shear coefficient depends on the beam cross-section profile, [15] and [16]. Stiffness properties for a complex thin-walled girder are determined by the strip element method [17].

The beam is loaded with transverse inertia load per unit length, and the distributed bending moment is expressed as

$$q_x = -m\frac{\partial^2 w}{\partial t^2}, \qquad m_x = J\frac{\partial^2 \psi}{\partial t^2}, \tag{2}$$

where $m = \rho A$ is the specific mass per unit length and $J = \rho I$ is its moment of inertia.

The equilibrium of moments and forces

$$\frac{\partial M}{\partial x} - Q = m_x, \qquad \frac{\partial Q}{\partial x} = -q_x \tag{3}$$

leads to two differential equations

$$D\frac{\partial^2 \psi}{\partial x^2} - S\left(\frac{\partial w}{\partial x} + \psi\right) - J\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{4}$$

$$S\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x}\right) - m\frac{\partial^2 w}{\partial t^2} = 0 \,. \tag{5}$$

From equation (5) one obtains

$$\frac{\partial \psi}{\partial x} = -\frac{\partial^2 w}{\partial x^2} + \frac{m}{S}\frac{\partial^2 w}{\partial t^2} \tag{6}$$

and by substituting (6) into (4) differentiated by x, one obtains the beam differential equation of motion

$$\frac{\partial^4 w}{\partial x^4} - \left(\frac{J}{D} + \frac{m}{S}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{m}{D}\frac{\partial^2}{\partial t^2}\left(w + \frac{J}{S}\frac{\partial^2 w}{\partial t^2}\right) = 0 . \tag{7}$$

Once (7) is solved, the angle of rotation is obtained from (6) as

$$\psi = -\frac{\partial w}{\partial x} + \frac{m}{S}\int \frac{\partial^2 w}{\partial t^2}\mathrm{d}x + f\left(t\right), \tag{8}$$

where $f\left(t\right)$ is the rigid body motion.

If w is extracted from (4) and substituted into (5), the same type of differential equation as (7) is obtained for ψ, and as (8) for w.

2.2 General solution to natural vibrations

In natural vibrations $w = W\sin\omega t$ and $\psi = \Psi\sin\omega t$, and Eqs. (7) and (8) are reduced to the vibration amplitudes

$$\frac{\mathrm{d}^4 W}{\mathrm{d}x^4} + \omega^2\left(\frac{J}{D} + \frac{m}{S}\right)\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} + \omega^2\frac{m}{D}\left(\omega^2\frac{J}{S} - 1\right)W = 0 \tag{9}$$

$$\Psi = -\frac{\mathrm{d}W}{\mathrm{d}x} - \omega^2\frac{m}{S}\int W\mathrm{d}x + C . \tag{10}$$

A solution to (9) can be assumed in the form $W = Ae^{\gamma x}$ that leads to a biquadratic equation

$$\gamma^4 + a\gamma^2 + b = 0 , \tag{11}$$

where

$$a = \omega^2\left(\frac{J}{D} + \frac{m}{S}\right), \qquad b = \omega^2\frac{m}{D}\left(\omega^2\frac{J}{S} - 1\right). \tag{12}$$

Roots of (11) read

$$\gamma = \alpha, -\alpha, i\beta, -i\beta , \tag{13}$$

where $i = \sqrt{-1}$ and

$$\alpha = \frac{\omega}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{m}{S} - \frac{J}{D}\right)^2 + \frac{4m}{D\omega^2}} - \left(\frac{m}{S} + \frac{J}{D}\right)} \tag{14}$$

$$\beta = \frac{\omega}{\sqrt{2}}\sqrt{\sqrt{\left(\frac{m}{S} - \frac{J}{D}\right)^2 + \frac{4m}{D\omega^2}} + \left(\frac{m}{S} + \frac{J}{D}\right)} . \tag{15}$$

Deflection function with its derivatives and the first integral can be presented in the matrix form

$$\begin{Bmatrix} W \\ W' \\ W'' \\ W''' \\ \int W\mathrm{d}x \end{Bmatrix} = \begin{bmatrix} sh\alpha x & ch\alpha x & \sin\beta x & \cos\beta x \\ \alpha ch\alpha x & \alpha sh\alpha x & \beta\cos\beta x & -\beta\sin\beta x \\ \alpha^2 sh\alpha x & \alpha^2 ch\alpha x & -\beta^2\sin\beta x & -\beta^2\cos\beta x \\ \alpha^3 ch\alpha x & \alpha^3 sh\alpha x & -\beta^3\cos\beta x & \beta^3\sin\beta x \\ \frac{1}{\alpha}ch\alpha x & \frac{1}{\alpha}sh\alpha x & -\frac{1}{\beta}\cos\beta x & \frac{1}{\beta}\sin\beta x \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}. \tag{16}$$

According to the solution to equations (9), (10) and (1), beam displacements and forces read

$$W = A_1 sh\alpha x + A_2 ch\alpha x + A_3\sin\beta x + A_4\cos\beta x \tag{17}$$

$$\begin{aligned} \Psi = -\Bigg[& \left(1+\frac{\omega^2 m}{\alpha^2 S}\right)\alpha A_1 ch\alpha x + \left(1+\frac{\omega^2 m}{\alpha^2 S}\right)\alpha A_2 sh\alpha x \\ & + \left(1-\frac{\omega^2 m}{\beta^2 S}\right)\beta A_3\cos\beta x - \left(1-\frac{\omega^2 m}{\beta^2 S}\right)\beta A_4\sin\beta x \Bigg] \end{aligned} \tag{18}$$

$$M = -D\left[\left(\alpha^2+\omega^2\frac{m}{S}\right)\left(A_1 sh\alpha x + A_2 ch\alpha x\right) - \left(\beta^2-\omega^2\frac{m}{S}\right)\left(A_3\sin\beta x + A_4\cos\beta x\right)\right] \tag{19}$$

$$Q = -\frac{\omega^2 m}{\alpha\beta}\left(\beta A_1 ch\alpha x + \beta A_2 sh\alpha x - \alpha A_3\cos\beta x + \alpha A_4\sin\beta x\right). \tag{20}$$

Relative values of constants A_i, $i = 1,2,3,4,$ are determined by satisfying four boundary conditions. Since there is no additional condition constant, C in (10) is ignored.

Coefficient α, Eq. (14), can be zero, in which case $\omega_0 = \sqrt{S/J}$ and $\beta_0 = \sqrt{(S/D)+(m/J)}$. Deflection function according to (17) takes the form

$$W = A_1 x + A_2 + A_3\sin\beta_0 x + A_4\cos\beta_0 x, \tag{21}$$

where the first two terms describe the rigid body motion. If $\omega > \omega_0$, then $\alpha = i\tilde{\alpha}$, where

$$\tilde{\alpha} = \frac{\omega}{\sqrt{2}}\sqrt{\left(\frac{m}{S}+\frac{J}{D}\right) - \sqrt{\left(\frac{m}{S}-\frac{J}{D}\right)^2 + \frac{4m}{D\omega^2}}} \tag{22}$$

and the deflection function reads

$$W = A_1\sin\tilde{\alpha}x + A_2\cos\tilde{\alpha}x + A_3\sin\beta x + A_4\cos\beta x\,. \tag{23}$$

Expressions for displacements and forces, Eqs (17-20), have to be transformed accordingly. Hence, $ch\alpha x = \cos\tilde{\alpha}x$, $sh\alpha x = i\sin\tilde{\alpha}x$ (an imaginary unit is included in constant A_1), $\alpha^2 = -\tilde{\alpha}^2$, instead of a single factor α it is necessary to write $\tilde{\alpha}$, and finally all functions associated with A_1 and A_2 must have the same sign as those associated with A_3 and A_4.

The above analysis shows that the beam has a lower and higher frequency spectral response, and a transition one. Frequency spectra are shifted for the threshold frequency ω_0. This problem is also investigated in [2,18]. The basic differential equations (4) and (5) are solved in [19] by assuming a solution in the form $w = Ae^{\gamma x}$ and $\psi = Be^{\gamma x}$, and the same expressions for displacements (17) and (18) are obtained.

3. Modified beam theory

3.1 Differential equations of motion

Beam deflection w and the angle of rotation ψ are split into their constitutive parts, Fig. 1,

$$w = w_b + w_s, \qquad \psi = \varphi + \vartheta, \qquad \varphi = -\frac{\partial w_b}{\partial x}, \tag{24}$$

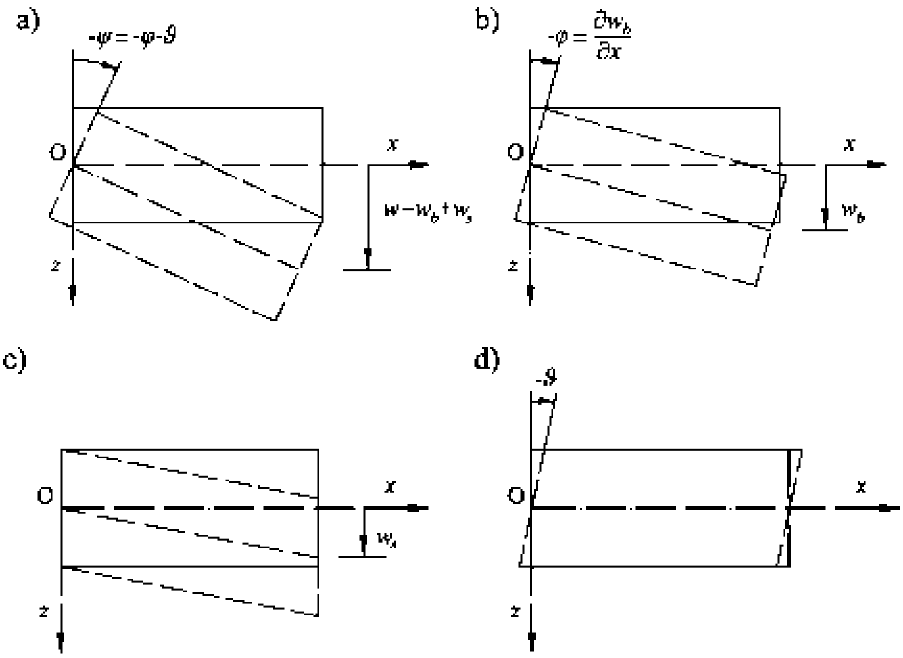

Fig. 1 Thick beam displacements: a – total deflection and rotation w, ψ, b – pure bending deflection and rotation w_b, φ, c – transverse shear deflection w_s, d – axial shear angle ϑ

where w_b and w_s are the beam deflections due to pure bending and transverse shear, respectively, and φ is the angle of cross-section rotation due to bending, while ϑ is the cross-section slope due to axial shear. Equilibrium equations (4) and (5) can be presented in the form with separated variables w_b and w_s, and ϑ

$$D\frac{\partial^3 w_b}{\partial x^3} - J\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_b}{\partial x}\right) + S\frac{\partial w_s}{\partial x} = D\frac{\partial^2 \vartheta}{\partial x^2} - S\vartheta - J\frac{\partial^2 \vartheta}{\partial t^2} \tag{25}$$

$$S\frac{\partial^2 w_s}{\partial x^2} - m\frac{\partial^2}{\partial t^2}\left(w_b + w_s\right) = -S\frac{\partial \vartheta}{\partial x}. \tag{26}$$

Since only two equations are available for three variables, one can assume that flexural and axial shear displacement fields are not coupled. In that case, by setting both the left and the right hand side of (25) to zero, it follows that

$$w_s = -\frac{D}{S}\frac{\partial^2 w_b}{\partial x^2} + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}. \tag{27}$$

By substituting (27) into (26) differential equation for flexural vibrations is obtained, which is expressed by a pure bending deflection

$$\frac{\partial^4 w_b}{\partial x^4} - \left(\frac{J}{D} + \frac{m}{S}\right)\frac{\partial^4 w_b}{\partial x^2 \partial t^2} + \frac{m}{D}\frac{\partial^2}{\partial t^2}\left(w_b + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}\right) = \frac{S}{D}\frac{\partial \vartheta}{\partial x}. \tag{28}$$

Disturbing function on the right hand side in (28) can be ignored due to the assumed uncoupling. Once w_b is determined, the total beam deflection, according to (24), reads

$$w = w_b - \frac{D}{S}\frac{\partial^2 w_b}{\partial x^2} + \frac{J}{S}\frac{\partial^2 w_b}{\partial t^2}. \tag{29}$$

The right hand side of (25) represents a differential equation of axial shear vibrations

$$\frac{\partial^2 \vartheta}{\partial x^2} - \frac{S}{D}\vartheta - \frac{J}{D}\frac{\partial^2 \vartheta}{\partial t^2} = 0. \tag{30}$$

3.2 General solution to flexural natural vibrations

Natural vibrations are harmonic, i.e. $w_b = W_b \sin \omega t$ and $\vartheta = \Theta \sin \omega t$, so that equations of motion (28) and (30) are related to the vibration amplitudes

$$\frac{\mathrm{d}^4 W_b}{\mathrm{d}x^4} + \omega^2\left(\frac{J}{D} + \frac{m}{S}\right)\frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2} + \omega^2\frac{m}{D}\left(\omega^2\frac{J}{S} - 1\right)W_b = 0 \tag{31}$$

$$\frac{\mathrm{d}^2 \Theta}{\mathrm{d}x^2} + \frac{S}{D}\left(\omega^2\frac{J}{S} - 1\right)\Theta = 0. \tag{32}$$

The amplitude of total deflection, according to (29), reads

$$W = \left(1 - \omega^2\frac{J}{S}\right)W_b - \frac{D}{S}\frac{\mathrm{d}^2 W_b}{\mathrm{d}x^2}. \tag{33}$$

Eq. (31) is known in literature as an approximate alternative of Timoshenko's differential equations, [3,20].

By comparing (31) with (9), it is obvious that the differential equation of flexural vibrations of the modified beam theory is of the same structure as that of the Timoshenko beam theory, but they are related to different variables, i.e. to W and W_b deflection, respectively. Therefore, the general solution for W presented in Section 2.2 is valid for W_b with all derivatives. In that case, flexural displacements and sectional forces read

$$\begin{aligned} W = {} & B_1\left(1 - \omega^2\frac{J}{S} - \alpha^2\frac{D}{S}\right)sh\alpha x + B_2\left(1 - \omega^2\frac{J}{S} - \alpha^2\frac{D}{S}\right)ch\alpha x \\ & + B_3\left(1 - \omega^2\frac{J}{S} + \beta^2\frac{D}{S}\right)\sin\beta x + B_4\left(1 - \omega^2\frac{J}{S} + \beta^2\frac{D}{S}\right)\cos\beta x \end{aligned} \tag{34}$$

$$\Phi = -\frac{\mathrm{d}W_b}{\mathrm{d}x} = -\left(B_1\alpha ch\alpha x + B_2\alpha sh\alpha x + B_3\beta\cos\beta x - B_4\beta\sin\beta x\right) \tag{35}$$

$$M=-D\frac{\mathrm{d}^2W_b}{\mathrm{d}x^2}=-D\left(B_1\alpha^2 sh\alpha x+B_2\alpha^2 ch\alpha x-B_3\beta^2\sin\beta x-B_4\beta^2\cos\beta x\right) \tag{36}$$

$$\begin{aligned}Q=-D\frac{\mathrm{d}^3W_b}{\mathrm{d}x^3}-\omega^2 J\frac{\mathrm{d}W_b}{\mathrm{d}x}=-D\Bigg[&B_1\alpha\left(\alpha^2+\omega^2\frac{J}{D}\right)ch\alpha x+B_2\alpha\left(\alpha^2+\omega^2\frac{J}{D}\right)sh\alpha x-\\&-B_3\beta\left(\beta^2-\omega^2\frac{J}{D}\right)\cos\beta x+B_4\beta\left(\beta^2-\omega^2\frac{J}{D}\right)\sin\beta x\Bigg].\end{aligned} \tag{37}$$

Parameters α and β are specified in Section 2.2, Eqs (14) and (15), respectively.

In this case, parameter α can also be zero, which gives $\omega_0=\sqrt{S/J}$ and $\beta_0=\sqrt{(S/D)+(m/J)}$. By taking this fact into account, the bending deflection W_b is of the form (21), while the total deflection according to (43), reads

$$W=B_1^*x+B_2^*+\frac{D}{S}\beta_0^2\left(B_3\sin\beta_0 x+B_4\cos\beta_0 x\right), \tag{38}$$

where B_1^* and B_2^* are the new integration constants instead of B_1 and B_2, which are infinite due to zero coefficients.

Concerning the higher order frequency spectrum, the governing expressions for displacements and forces, Eqs. (34-37), have to be transformed in the same manner as explained in Section 2.2.

3.3 General solution to axial shear natural vibrations

Differential equation (32) for natural axial shear vibrations of beam reads

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}+\left(\omega^2\frac{J}{D}-\frac{S}{D}\right)\Theta=0\,. \tag{39}$$

It is similar to the equation for rod stretching vibrations

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\omega_R^2\frac{m}{EA}u=0\,. \tag{40}$$

The difference is in the additional moment $S\Theta$ which is associated to the inertia moment $\omega^2 J\Theta$ and represents the reaction of an imaginary rotational elastic foundation with stiffness equal to the shear stiffness S, as shown in Fig. 2.

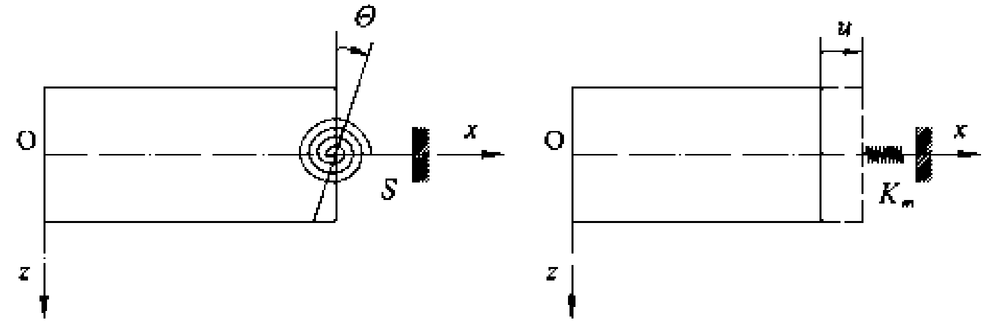

Fig. 2 Analogy between an axial shear model and a stretching model

Solution to (40) and a corresponding axial force $N = EA\frac{\mathrm{d}u}{\mathrm{d}x}$ read

$$u = C_1 \sin \chi x + C_2 \cos \chi x \tag{41}$$

$$N = EA\left(C_1 \chi \cos \chi x - C_2 \chi \sin \chi x\right), \tag{42}$$

where $\chi = \omega_R \sqrt{m/(EA)}$. Based on the analogy between (39) and (40), for the shear slope angle and moment one can write

$$\Theta = C_1 \sin \eta x + C_2 \cos \eta x \tag{43}$$

$$M = D\left(C_1 \eta \cos \eta x - C_2 \eta \sin \eta x\right), \tag{44}$$

where

$$\eta = \sqrt{\omega^2 \frac{J}{D} - \frac{S}{D}}. \tag{45}$$

Between the natural frequencies of axial shear beam vibrations and stretching vibrations there is a relation $\omega^2 = \omega_0^2 + \omega_R^2$, where $\omega_0 = \sqrt{S/J}$ belongs to the axial shear mode obtained from (39), $\Theta = A_0 + A_1 x$, (which reminds us of a sheared set of playing cards). It is interesting that ω_0 is at the same time the threshold frequency of flexural vibrations, as explained in Section 2.2.

4. Comparison between the Timoshenko beam theory and the new theory

4.1 Dynamic response

As elaborated in Section 2.1, the Timoshenko beam theory deals with two differential equations of motion with two basic variables, i.e. deflection and the angle of rotation. That system is reduced to a single equation in terms of deflection and all physical quantities depend on its solution. On the other hand, in the modified beam theory, Section 3.1, the total deflection is divided into the pure bending deflection and the shear deflection, while the total angle of rotation consists of pure bending rotation and axial shear angle. The governing equations are condensed into a single one for flexural vibrations with the bending deflection as the main variable and another variable for axial shear vibrations. Differential equations for flexural vibrations in both theories are of the same structure, Eqs. (9) and (31), resulting in the same hyperbolic and trigonometric functions in the solution for W and W_b. However, expressions for displacements and forces are different due to different coefficients associated to the integration constants.

In order to obtain the same expressions for displacements and forces in both theories, the following relations between the constants, based on identical deflections, Eqs. (17) and (34), should exist

$$A_i = \left(1 - \omega^2 \frac{J}{S} - \alpha^2 \frac{D}{S}\right) B_i, \quad i = 1,2 \tag{46}$$

$$A_i = \left(1 - \omega^2 \frac{J}{S} + \beta^2 \frac{D}{S}\right) B_i, \quad i = 3,4. \tag{47}$$

Indeed, if relations (46) and (47) are substituted into Eqs. (18), (19) and (20), and if expressions (14) and (15) for α and β are taken into account, expressions (18), (19), and

(20) become identical to (35), (36), and (37). For illustration, let us check the identity of the first terms in expressions for shear forces, Eqs. (20) and (37)

$$\frac{\omega^2 m}{\alpha} A_1 = D\alpha\left(\alpha^2 + \omega^2 \frac{J}{D}\right) B_1 . \tag{48}$$

By taking (46) into account, (48) can be presented in the form

$$\omega^2 m\left(1 - \omega^2 \frac{J}{S} - \alpha^2 \frac{D}{S}\right) = \alpha^2\left(\alpha^2 + \omega^2 \frac{J}{D}\right). \tag{49}$$

If (14) is substituted for α, relation (49) is satisfied.

Based on the above fact, flexural vibrations determined by the Timoshenko beam theory and its modification are identical. Therefore, axial shear vibrations, extracted from the Timoshenko beam theory, are not coupled with flexural vibrations, as assumed in Section 3.1.

4.2 Static response

One expects that expressions for static displacements can be obtained straightforwardly by deducting dynamic expressions. In the case of the Timoshenko beam theory, the static term of Eq. (9) leads to $W = A_0 + A_1 x + A_2 x^2 + A_3 x^3$, and Eq. (10) gives $\Psi = -\left(A_1 + 2A_2 x + 3A_3 x^3\right)$. That results in the zero shear force Q, Eq. (1), which is also obvious from (20) if $\omega = 0$ is taken into account. Therefore, in order to overcome this problem, it is necessary to return back to Eqs (4) and (5) with static terms. By substituting (5) into (4), $D\mathrm{d}^3\Psi / \mathrm{d}x^3 = 0$, i.e. $\Psi = -\left(A_1 + 2A_2 x + 3A_3 x^3\right)$ is obtained. Based on the known Ψ, one obtains from (4)

$$W = \frac{D}{S}\frac{\mathrm{d}\Psi}{\mathrm{d}x} - \int \Psi \mathrm{d}x + A_0 = A_0 + A_1 x + A_2 x^2 + A_3 x^3 - \frac{2D}{S}\left(A_2 + 3A_3 x\right). \tag{50}$$

On the other hand, the static part of Eq. (31) of the modified beam theory gives $W_b = B_0 + B_1 x + B_2 x^2 + B_3 x^3$, and from (33) it follows that

$$W = W_b - \frac{D}{S}\frac{\mathrm{d}^2 W_b}{\mathrm{d}x} = B_0 + B_1 x + B_2 x^2 + B_3 x^3 - \frac{2D}{S}\left(B_2 + 3B_3 x\right), \tag{51}$$

which is an expression identical to (50). The angle of rotation is $\Phi = -\mathrm{d}W_b / \mathrm{d}x = -\left(B_1 + 2B_2 x + 3B_3 x^2\right)$, which is the same as the above Ψ in the Timoshenko beam theory.

5. Beam finite element based on the modified beam theory

The Timoshenko beam theory deals with two variables of flexural vibrations, W and Ψ, which are of the same importance. Therefore, in the development of the beam finite element, one takes into account the independent shape functions for W and Ψ of the same order. That leads to the shear locking problem, which is remedied by an additional degree of freedom. That problem can be avoided if the static solution for W and Ψ from Section 4.2, which includes relation (10), is used for the shape functions [13]. In that case, one obtains the same beam finite element properties as for the modified beam theory. Derivation of the finite element by the latter theory is simpler and more transparent and that is why it is used here.

A relatively simple two-node beam finite element can be derived in an ordinary way if the static solution is used for deflection interpolation functions, Section 4.2

$$W_b = a_0 + a_1\frac{x}{l} + a_2\left(\frac{x}{l}\right)^2 + a_3\left(\frac{x}{l}\right)^3 \tag{52}$$

$$W_s = -2\varepsilon\left(a_2 + 3a_3\frac{x}{l}\right) \tag{53}$$

$$W = a_0 + a_1\frac{x}{l} + a_2\left(\frac{x}{l}\right)^2 + a_3\left(\frac{x}{l}\right)^3 - 2\varepsilon\left(a_2 + 3a_3\frac{x}{l}\right) \tag{54}$$

$$\varPhi = -\frac{\mathrm{d}W_b}{\mathrm{d}x} = -\frac{1}{l}\left[a_1 + 2a_2\frac{x}{l} + 3a_3\left(\frac{x}{l}\right)^2\right], \tag{55}$$

where $\varepsilon = D/\left(Sl^2\right)$ and l is the element length. By satisfying alternatively the unit value for one of the nodal displacements and the zero value for the remaining displacements, one can write

$$\{\delta\} = [C]\{A\}, \tag{56}$$

where

$$\{\delta\} = \begin{Bmatrix} W_1 \\ -\varPhi_1 \\ W_2 \\ -\varPhi_2 \end{Bmatrix}, \quad \{A\} = \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix}, \tag{57}$$

$$[C] = \begin{bmatrix} 1 & 0 & -2\varepsilon & 0 \\ 0 & \frac{1}{l} & 0 & 0 \\ 1 & 1 & 1-2\varepsilon & 1-6\varepsilon \\ 0 & \frac{1}{l} & \frac{2}{l} & \frac{3}{l} \end{bmatrix}. \tag{58}$$

Inversion of (56) is

$$\{A\} = [C]^{-1}\{\delta\}, \tag{59}$$

where

$$[C]^{-1} = \frac{1}{1+12\varepsilon}\begin{bmatrix} 1+6\varepsilon & -4\varepsilon\left(1+3\varepsilon\right)l & 6\varepsilon & -2\varepsilon\left(1-6\varepsilon\right)l \\ 0 & \left(1+12\varepsilon\right)l & 0 & 0 \\ -3 & -2\left(1+3\varepsilon\right)l & 3 & -\left(1-6\varepsilon\right)l \\ 2 & l & -2 & l \end{bmatrix}. \tag{60}$$

Bending deflection (52), by employing (59), yields

$$W_b = \langle P\rangle_b\{A\} = \langle f\rangle_b\{\delta\}, \tag{61}$$

where $\langle P\rangle_b = \langle 1\ x/l\ (x/l)^2\ (x/l)^3\rangle$ and

$$\langle f\rangle_b = \langle P\rangle_b [C]^{-1} \tag{62}$$

is the vector of bending shape functions. Referring to the finite element method [21,22], the bending stiffness matrix is defined as

$$[K]_b = D\int_0^l \left\{\frac{\mathrm{d}^2 f_b}{\mathrm{d}x^2}\right\}\left\langle\frac{\mathrm{d}^2 f_b}{\mathrm{d}x^2}\right\rangle \mathrm{d}x = \frac{4D}{l^4}[C]^{-T}\int_0^l \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 3\dfrac{x}{l}\end{Bmatrix}\left\langle 0\ 0\ 1\ 3\frac{x}{l}\right\rangle \mathrm{d}x [C]^{-1}, \tag{63}$$

where symbolically $[C]^{-T} = \left([C]^{-1}\right)^T$. After integration and multiplication one obtains

$$[K]_b = \frac{2D}{(1+12\varepsilon)^2 l^2}\begin{bmatrix} 6 & 3l & -6 & 3l \\ & 2\left[1+6\varepsilon(1+6\varepsilon)\right]l^2 & -3l & \left[1-12\varepsilon(1+6\varepsilon)\right]l^2 \\ & & 6 & -3l \\ Sym. & & & 2\left[1+6\varepsilon(1+6\varepsilon)\right]l^2 \end{bmatrix}. \tag{64}$$

In a similar way, shear deflection (53) can be presented in the form

$$W_s = \langle P\rangle_s \{A\} = \langle f\rangle_s \{\delta\}, \tag{65}$$

where $\langle P\rangle_s = -2\varepsilon\langle 0\ 0\ 1\ 3x/l\rangle$ and

$$\langle f\rangle_s = \langle P\rangle_s [C]^{-1} \tag{66}$$

is the vector of shear shape functions. The shear stiffness matrix reads [22]

$$[K]_s = S\int_0^l \left\{\frac{\mathrm{d}f_s}{\mathrm{d}x}\right\}\left\langle\frac{\mathrm{d}f_s}{\mathrm{d}x}\right\rangle \mathrm{d}x = S\left(\frac{6\varepsilon}{l}\right)^2 [C]^{-T}\int_0^l \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1\end{Bmatrix}\langle 0\ 0\ 0\ 1\rangle \mathrm{d}x [C]^{-1}. \tag{67}$$

That leads to

$$[K]_s = \frac{36\varepsilon^2 S}{(1+12\varepsilon)^2 l}\begin{bmatrix} 4 & 2l & -4 & 2l \\ & l^2 & -2l & l^2 \\ & & 4 & -2l \\ Sym. & & & l^2 \end{bmatrix}. \tag{68}$$

Bending and shear stiffness matrices can be summed and the total stiffness matrix reads

$$\left[K\right]=\frac{2D}{\left(1+12\varepsilon\right)^{2}l}\begin{bmatrix}6 & 3l & -6 & 3l\\ & 2\left(1+3\varepsilon\right)l^{2} & -3l & \left(1-6\varepsilon\right)l^{2}\\ & & 6 & -3l\\ Sym. & & & 2\left(1+3\varepsilon\right)l^{2}\end{bmatrix}. \tag{69}$$

Mass matrix, according to definition [20] is

$$\left[M\right]_{m}=m\int_{0}^{l}\left\{f_{w}\right\}\left\langle f_{w}\right\rangle \mathrm{d}x\,, \tag{70}$$

where

$$\left\langle f\right\rangle_{w}=\left\langle P\right\rangle_{w}\left[C\right]^{-1}, \tag{71}$$

is the shape function of total deflection, and $\left\langle P\right\rangle_{w}=\left\langle 1\;\; x/l\;\; \left[\left(x/l\right)^{2}-2\varepsilon\right]\;\; \left[\left(x/l\right)^{2}-6\varepsilon\right]x/l\right\rangle$. By substituting (71) into (70) and after integration, one obtains

$$\left[M\right]_{m}=\frac{ml}{420\left(1+12\varepsilon\right)^{2}}\times$$
$$\begin{bmatrix}156+3528\varepsilon+20160\varepsilon^{2} & \left(22+462\varepsilon+2520\varepsilon^{2}\right)l & 54+1512\varepsilon+10080\varepsilon^{2} & -\left(13+378\varepsilon+2520\varepsilon^{2}\right)l\\ & \left(4+84\varepsilon+504\varepsilon^{2}\right)l^{2} & \left(13+378\varepsilon+2520\varepsilon^{2}\right)l & -\left(3+84\varepsilon+504\varepsilon^{2}\right)l^{2}\\ & & 156+3528\varepsilon+20160\varepsilon^{2} & -\left(22+462\varepsilon+2520\varepsilon^{2}\right)l\\ Sym. & & & \left(4+84\varepsilon+504\varepsilon^{2}\right)l^{2}\end{bmatrix}. \tag{72}$$

In a similar way, according to definition [20], one finds the mass moment of inertia matrix

$$\left[M\right]_{J}=J\int_{0}^{l}\left\{\frac{\mathrm{d}f_{b}}{\mathrm{d}x}\right\}\left\langle\frac{\mathrm{d}f_{b}}{\mathrm{d}x}\right\rangle \mathrm{d}x$$
$$=\frac{J}{30\left(1+12\varepsilon\right)^{2}l}\begin{bmatrix}36 & \left(3-180\varepsilon\right)l & -36 & \left(3-180\varepsilon\right)l\\ & \left(4+60\varepsilon+1440\varepsilon^{2}\right)l^{2} & \left(-3+180\varepsilon\right)l & -\left(1+60\varepsilon-720\varepsilon^{2}\right)l^{2}\\ & & 36 & \left(-3+180\varepsilon\right)l\\ Sym. & & & \left(4+60\varepsilon+1440\varepsilon^{2}\right)l^{2}\end{bmatrix}. \tag{73}$$

Beam axial shear vibrations are analogous to stretching vibrations, Section 3.3, and the vector of shape functions is $\left\langle f\right\rangle_{a}=\left\langle 1-x/l\;\; x/l\right\rangle$. The following stiffness and mass matrices are obtained

$$\left[K\right]_{a}=\frac{D}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+\frac{Sl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}, \tag{74}$$

$$\left[M\right]_{a}=J\frac{l}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}. \tag{75}$$

6. Numerical examples

In order to evaluate the developed finite element, flexural vibrations of a simply supported, clamped and free beam are analysed and compared with analytical solutions [23] and 2D FEM results obtained by NASTRAN [24]. The beam length is $L = 10$ m and height $H = 2$ m. The 1D FEM model includes 50 beam elements and the 2D model 50x6=300 membrane elements.

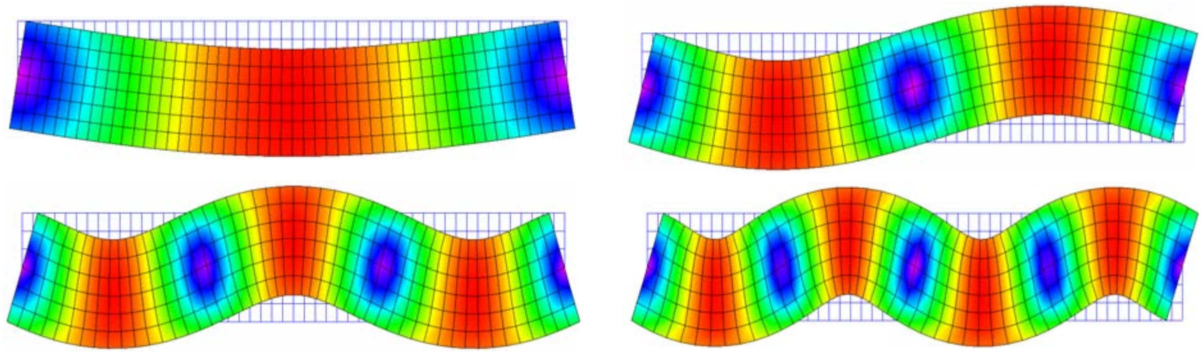

Fig. 3 The first four natural modes of a simply supported beam

Table 1 presents the obtained values of the frequency parameter $\lambda = \omega / \omega_0$ for the simply supported beam, where $\omega_0 = \sqrt{S / J}$ is the threshold frequency obtained from the last term in differential equation (31). It is well known that the simply supported beam exhibits a double frequency spectrum for the same mode shapes shown in Fig. 3 [2,18]. 1D FEM results follow very well the analytical solutions for the first spectrum up to the threshold frequency. 2D FEM results agree very well with the first spectrum of analytical solution. However, 1D and 2D FEM analyses cannot capture the second frequency spectrum.

Table 1 Frequency parameter $\lambda = \omega / \omega_0$ for a simply supported beam, $h / l = 0.2$, $k = 5 / 6$

n	Analytical		FEM	
	1st spectrum, λ_n^{f1}	2nd spectrum, λ_n^{f2}	1D, $\lambda_n^{(1)}$	2D, $\lambda_n^{(2)}$
0		1.000*		
1	0.055	1.064	0.052	0.055
2	0.189	1.227	0.183	0.191
3	0.362	1.445	0.351	0.366
4	0.549	1.693	0.535	0.558
5	0.741	1.959	0.724	0.755
6	0.935	2.237	0.916	0.953
6.335*	1.000*			
7	1.128	2.524	1.064	1.151
8	1.321	2.816	1.108	1.346
9	1.512	3.113	1.223	1.538
10	1.702	3.414	1.301	1.727

*Threshold

Values of frequency parameter for the clamped and the free beam are compared in Table 2 and 3, respectively. Very good agreement between the 1D and 2D FEM results on the one hand and the analytical solutions on the other are shown.

Table 2 Frequency parameter $\lambda = \omega / \omega_0$ for a clamped beam, $h / l = 0.2$, $k = 5/6$

Mode no. j	Analytical, λ_j	FEM	
		1D, $\lambda_j^{(1)}$	2D, $\lambda_j^{(2)}$
1	0.106	0.102	0.107
2	0.242	0.236	0.247
3	0.404	0.394	0.412
4	0.577	0.564	0.590
5	0.758	0.742	0.775
6	0.941	0.923	0.960
*	1.000*		
7	1.066	1.065	1.047
8	1.123	1.105	1.139
9	1.235	1.229	1.211
10	1.314	1.295	1.331

*Threshold

Table 3 Frequency parameter $\lambda = \omega / \omega_0$ for a free beam, $h / l = 0.2$, $k = 5/6$

Mode no. j	Analytical, λ_j	FEM	
		1D, $\lambda_j^{(1)}$	2D, $\lambda_j^{(2)}$
1	0.117	0.112	0.116
2	0.272	0.264	0.273
3	0.453	0.441	0.455
4	0.638	0.623	0.642
5	0.819	0.803	0.825
6	0.967	0.956	0.967
*	1.000*		
7	1.070	1.075	1.071
8	1.097	1.087	1.094
9	1.272	1.260	1.263
10	1.279	1.265	1.269

*Threshold

Table 4 shows the frequency parameter of axial shear vibrations. The finite element developed in Section 5, Eqs. (74) and (75), gives very reliable results.

Table 4 Frequency parameter $\lambda = \omega / \omega_0$ for axial shear vibrations, $h/l = 0.2$, $k = 5/6$

n	Analytical, λ_n^S	1D FEM, λ_n^{S1}
0	1.000*	
1	1.050	1.050
2	1.188	1.188
3	1.387	1.388
4	1.625	1.628
5	1.888	1.894
6	2.167	2.177
7	2.455	2.472
8	2.751	2.776
9	3.052	3.088
10	3.356	3.407

*Threshold

7. Conclusion

The Timoshenko beam theory deals with the total deflection and the cross-section rotation as two basic variables. The modified beam theory is an extension of the former from flexural to axial shear vibrations. The main variables are the pure bending deflection and the axial shear slope angle. The modified flexural beam theory is known in literature as an approximate variant of the Timoshenko theory. By a linear transformation of expressions for displacements and sectional forces it is shown that the modified theory is exact as the Timoshenko theory.

The developed sophisticated beam finite element based on the modified and extended Timoshenko beam theory gives very good results, the same as the 2D FEM analysis, when compared to the analytical solutions.

ACKNOWLEDGMENT

This study was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean Government (MEST) through GCRC-SOP (Grant No. 2011-0030669).

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Submitted: 11.4.2013

Accepted: 22.11.2013

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